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Mergers in symmetric and asymmetric noncooperative auction markets: the effects on prices and efficiency

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Abstract

United States antitrust merger analysis has recently focused on simulating the unilateral effects of mergers. We develop a model to simulate the unilateral price increase from a merger in an auction market. We illustrate our results in the context of hospital mergers in the U.S., and calibrate our simulations to known market parameters.

We compare the price increases in our model to those suggested by analytically simpler models. The simulation results suggest that the unilateral price increases predicted by our model are modest in general. We also simulate the merger cost savings that are needed to offset the price effects.

Keywords: Antitrust; Auctions; Bidding; Calibration; Mergers; Simulation; Unilateral

JEL classification: D44-auctions; L41-horizontal anticompetitive practices

1. Introduction

We provide a model for the evaluation of mergers of firms in auction markets (markets with bids for a fixed quantity). We focus only on such markets for which bid rigging (price fixing) is not likely. In game theoretic terms, we look at noncooperative Nash equilibria in prices; in antitrust language we look for the ‘‘unilateral effects’’ of a merger. Our model can be calibrated to provide insights

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into the price effects of mergers in markets for which some premerger parameter(*s*) are observable, whereas post merger parameters are not (premerger screening). While various papers look at related issues, Dalkir (1995) and Tschantz et al. (1997) are the only works that numerically simulate mergers under asymmetric first-price auctions with continuous cost distributions. Our methodology is related to the analysis of coalitions in first-price auction markets by Marshall et al. (1994). Waehrer (1997); Waehrer and Perry (1998); Thomas (1998) and Lebrun (1997) also present related analytical work.

Tschantz et al. (1997) compute the equilibrium pricing functions in a first-price asymmetric auction and compare the merger effects in first-price and second-price auctions; they report that the first-price asymmetric equilibrium inverse bidding functions have a singular critical value above which no bids are made. They use a distributional assumption that enables them to change the expected value of the cost signals[1] for a subset of players while holding their variance constant.[2] While the uniform distribution that we use does not have this property, it yields equilibrium pricing functions that are well-behaved, and simplifies the analytics as well as the computation. The uniform distribution also represents a ''worst case scenario,'' since any other distribution with thinner tails would result in a smaller percentage price increase, our results are likely upper bounds on the merger price effects from other types of distributions (holding the coefficient of variation constant across distributions).

We provide a proof of equilibrium existence in the auction; analyze efficiency implications of mergers; examine cases in which merger leads to symmetry; and calibrate the model to demonstrate that explicit modeling of asymmetries is required for ''realistic'' predictions from such models.[3]

We give some background and motivation in Section 2, present the model in Section 3, and present calibrated price and cost simulations in Section 4. Section 5 focuses on compensating efficiencies. In Sections 6 and 7 we discuss possible extensions and conclude.

2. Background

The emphasis on *unilateral effects* reflects concerns over mergers even when preconditions for coordination are not evident.[4] Such unilateral effects issues are

[1] Alternatively, utility signals in a high-price auction.

[2] Tschantz et al. (1997) remark that the price effect of a merger that increases the level of asymmetry in the market is directly proportional to the standard deviation of the firms' cost distribution. They use extreme value distributions. In contrast, Riley and Li (1997) use uniform and truncated normal distributions in their analysis of asymmetric auctions.

[3] Our model is applicable to other examples such as ''bidder preferences,'' where there are two types of bidders: subsidized and not subsidized, cf. Brannman and Froeb (1997).

[4] See Dalkir and Warren-Boulton (1998) for a recent case study of antitrust action involving estimation of unilateral effects.

now an important part of the U.S. Department of Justice and the Federal Trade Commission (1992) *Horizontal Merger Guidelines* (hereinafter ''Guidelines''), a document which describes how the government will address competitive issues in merger analysis.

As a subset of unilateral effect analysis, we estimate auction models to be potentially applicable to a third to a half of the mergers that concern the antitrust agencies.[5] Our model is derived for use in such analyses.[6] For a concrete application of the model, we discuss our results in the context of a single industry, hospitals. In hospital mergers, the U.S. antitrust agencies have been particularly concerned with the prices that will be paid in the ''bidding markets,'' such as the sale of services to Preferred Provider Organizations (PPOs) by hospitals.[7] A PPO provides health insurance coverage to individuals.[8] It achieves discounts from hospital list prices (e.g., 30 percent off list price) by asking local hospitals to bid for long term contracts for its entire clientele base, announcing prior to the auction that it will select only one (or two) local providers for its clientele. Since in a local market a single PPO may control a significant fraction of all potential patients, hospitals may face a substantial swing in their patient bases depending upon success in the auction.[9] This potential swing in patient base is a substantial

[5] Private conversations with Kenneth C. Baseman and Frederick R. Warren-Boulton. The latter served as the Deputy Assistant Attorney General for Economic Analysis, Antitrust Division, U.S. Department of Justice.

[6] To reemphasize, we are analyzing mergers in markets that can reasonably be thought of as auction markets, rather than just any market. In the real world, the size distribution of firms in any given industry is likely to be asymmetric (Gibrat's law predicts a lognormal distribution: Scherer, 1980, pp. 145–150), but the ordering of sizes may or may not be the same as the ordering of cost functions. In our model there is a positive relationship between efficiency and size, but this is true of all stylized models that assume noncooperative Nash profit-maximizing behavior. If two firms with nonidentical constant unit costs merge, we derive the cost distribution of the merger entity from the individual cost distributions of the merging firms' in a way that is consistent with one-period profit maximization. In our model, the most efficient (on average) firm is not necessarily the most likely acquirer of a less efficient firm; we treat the decision to merge, and the choice of a merger partner, as exogenous. In the real world likely the closest rivals would want to merge; but other characteristics play a significant role: a catholic hospital may be more likely to merge with another catholic hospital. The same is true for the timing of a merger, it probably involves many exogenous, even noneconomic, factors.

[7] When price discrimination is possible, the Guidelines (1992) state that the government will look at '' . . . markets consisting of a particular use or uses by groups of buyers . . . '' (Subsection 1.12).

[8] Aside from government insurance for the aged and some welfare clients, most hospitalization in the United States is provided through employer-provided insurance plans. These plans are of three primary types: indemnity insurance, PPO, and HMO. Under indemnity insurance, the insured can go to any hospital and receive a percentage reimbursement. In a PPO or HMO, the insured must go to *selected ''preferred'' providers* and typically has fully paid health care, though an HMO will have greater control over the services a patient receives. An HMO may use an auction, but more than price is important in its contracts. We therefore focus on PPO's here.

[9] We roughly characterize one actual case. Suppose a PPO has 25 percent of the local population facing 6 local hospitals. It announces that it will select two hospitals as ''winners.'' With symmetry, winning hospitals will each have 25 percent of the population, the losers will each have 12.5 percent.

incentive to win the bidding. The long term nature of the contracts and the fact that a single PPO controls a significant fraction of the patient base also complicates bid rigging; it is hard to compensate designated losers for agreeing to bid high.

Having described the market, we outline an introductory example on hospitals, from Baker (1997). Assume each hospital is separately owned with different costs, producing an indivisible output. For this example, assume all hospitals' costs are known by each bidder (but not by the buyer) and that the buyer wants to have k hospitals in its plan offering. Ordering hospitals, $1, \ldots, n$, from the lowest cost to the highest cost, there will be k winners, where $k < n$. The equilibrium bid will be the cost level of hospital $k+1$. If two of the first k hospitals merge, the merged hospital can demand an "all or nothing" bid price equal to the costs of hospital $k+2$.[10]

The assumption that each firm knows its rivals' costs for serving any specific buyer is very strong. In our context, a hospital's costs depend upon the demographics of, and vector of services demanded by, the members of any particular PPO (and one hospital may be better for a particular PPO). We apply the approach used in standard private-value auction theory. We assume that each firm knows its own cost and the distribution from which its rivals' costs are drawn.[11]

Following the antitrust analysis of Rule and Meyer (1990), we could treat hospitals as symmetric in what they refer to as "$1/N$ markets."[12] If we were to do so, at first glance it would appear that the calibration of an auction model could proceed as follows: (i) select a model of N symmetric profit maximizing bidders each with a cost drawn from a distribution; (ii) find the equilibrium to the auction game; (iii) calibrate the model to some known phenomena in the relevant market (e.g. the markup, or the mean and the range of unit costs); and (iv) using the parameters, simulate the merger by solving the identical model for the case of $N-1$ symmetric bidders.

What we demonstrate herein is that this methodology, albeit simple, is *highly* biased.[13] This is in part a consequence of the difference between the first order statistic on N draws from a cost distribution and the statistic from $N-1$ draws. This $1/N$ methodology implicitly assumes that in expectation the most efficient

[10] This argument is for one firm with multiple units. Masson et al. (1994) show that with known unit cost/value distributions for different units, oligopolists with multiple units may price some units "out of the market" (e.g., withhold units) in Bertrand–Nash equilibrium.

[11] Although explicitly the model assumes a homogeneous product, uncertainty in perceived quality can create the same types of behavior.

[12] Their concept relies on *a priori* equality of the marginal bids. E.g., a large firm may have cost advantages on infra marginal sales, but not on marginal sales.

[13] In the context of U.S. antitrust this is important. The Guidelines concentrate on price increases on the order of five percent. Furthermore, if one can show a small efficiency gain this may offset a small increase in margins (conditional upon no efficiency gains), leading to merger clearance. We address this in Section 5.

hospital post merger $[1/(N-1)]$ is less efficient than premerger $[1/N]$. As another simple alternative, one may use the best response functions for $N-1$ symmetric hospitals, applied to the order statistic from N cost draws. As we shall demonstrate, this too leads to a significant bias; the true best response functions lead to much lower expected prices.

While symmetric auctions are typically easily solved, asymmetric auction games are not.[14] But a merger is almost always a single bilateral agreement, creating asymmetry if a market is initially symmetric.[15] In our methodology, we start with N symmetric firms each with a cost draw and calibrate this model to find the markup predicted in the data. We then ''merge'' two firms in the sense of still modeling N cost draws, one draw by each of $N-2$ firms and two draws by the single merged firm.

The resulting $N-1$ firms take part in an asymmetric auction, which is not amenable to an analytic solution. Marshall et al. (1994) show that such a model can be solved using numerical methods, and provide a methodology for doing so. They interpret their results as a model of coalitions with equal rent sharing by each coalition participant. Independently we developed a different numerical methodology (cf., Dalkir, 1995) achieving similar solutions and additionally provide a proof that a unique equilibrium exists. We look explicitly at mergers in the noncooperative model, in which the rent to the initial partners can be negotiated over a range of Pareto-improving bargains.[16] Later we show some interesting properties of starting from asymmetric firms prior to merger.

In what follows we develop the basic model described above. We study both the price effects and efficiency effects which come from the violation of *revenue equivalence* in asymmetric auctions (cf., Maskin and Riley, 1996). In particular we demonstrate how strategic considerations may lead to some loss of efficiency if mergers make an initially symmetric auction asymmetric. We also examine markets which are initially asymmetric. In such cases mergers may lead to symmetry. In these cases we demonstrate that the strategic bidding effects of mergers may improve efficiency. In each case (symmetric to asymmetric or asymmetric to symmetric) we show that ignoring asymmetry (e.g., the $1/N$

[14] Asymmetry here refers to the asymmetry between firms: one firm has more cost draws than another and therefore it may adopt a different strategy when faced with the same bidding situation. This auction game is also one of asymmetric information; a firm knows its own cost but not the costs of its rivals. We use the terms symmetry and asymmetry for the former.

[15] In a calibrated auction market simulation of joint bidding for oil leases DeBrock and Smith (1983) start with 20 symmetric firms each with a single draw and contrast this with the symmetric case of 5 firms with 4 draws each, 4 with 5, etc. But, when simulating a merger, the symmetry assumption is very misleading.

[16] Marshall et al. (1994) focus on the stability of a cartel agreement with equal profit sharing, a free rider problem. We focus instead on Pareto-improving (to the firms) mergers which are legally binding on the merged entities.

treatment of mergers) can yield misleading predictions on prices and market efficiency.[17]

3. The model

Our methodology will be as follows. Suppose that each of $M \geqslant 3$ sellers has some number of cost draws from a technology density function and each can select to use its lowest cost draw. Premerger we define the number of draws by firm i as k_i. We model the merger of two firms, i and j, as a single firm with a total number of cost draws $k_{\text{merger}} = k_i + k_j$. If all firms have the same number of draws prior to merger, the post merger number of draws will differ among firms. This necessitates the analysis of asymmetric auctions. If asymmetric premerger, firms *may* be symmetric post merger.

We analyze a first price auction. Each seller will bid for the sale of its output, and will hereinafter be called a bidder. The i^{th} bidder has unit cost $c_i = \mu + \varepsilon_i$, where μ reflects a cost component that is common to all bidders and ε_i is a bidder-specific, random cost component independently and identically distributed (i.i.d.) uniformly over a common support $[-\Delta, \Delta]$. We assume $\mu \geqslant \Delta > 0$. One buyer (e.g., a PPO), purchases a fixed quantity of services, normalized to one, from the lowest bidder. As is customary in the literature, we assume that the buyer values the contract at $v = \mu + \Delta$ and that this is common knowledge. Under this assumption the range of potential equilibrium prices is $[\mu - \Delta, \mu + \Delta]$.

We start with the number of bidders equal to the number of i.i.d. cost draws, which we call N. Mergers are modeled by ‘‘regrouping’’ the N cost draws amongst M bidders where $M < N$. Now different firms may have different numbers of draws from the cost distribution, and the model becomes asymmetric. A firm with multiple cost draws is assumed to use the lowest cost technology from its set of draws.[18] For example in the typical two-firm merger, the merged firm with two cost draws faces $N - 2$ rivals each with a single cost draw.

To simplify the notation for the analytical work (though not the simulations), we normalize to $\mu = \frac{1}{2}$ and $\Delta = \frac{1}{2}$. From this model it is simple to rescale for calibration.

[17] The asymmetric models we numerically simulate include two types of bidders. While asymmetric auction models with more types of bidders may be interesting in themselves (cf. Froeb et al., 1997), our results are sufficient to demonstrate the implicit bias in symmetric approximations to asymmetric auction markets. Our calibration methodology is generalizable to an arbitrary number of types.

[18] For a ‘‘common value’’ auction, e.g., estimating the value of an oil pool as in DeBrock and Smith (1983), combining firms involves averaging signals; in this ‘‘private value’’ auction it involves selecting the most valuable ‘‘signal’’ or cost draw.

3.1. The symmetric model of the market

We review the symmetric case in detail to aid the exposition in the asymmetric case. The equilibrium best-response price function of any player given its cost can be written as

$$P_{[N]}(c_i) = \operatorname*{argmax}_{p_i} \pi_i(p_i|c_i, N).$$

In words, $P_{[N]}$ maximizes firm i's expected profits $\pi_i(p_i|c_i, N)$ with respect to its price, p_i, given its costs, c_i, and the number of firms, N. The notation $P_{[N]}$ is the equilibrium best response (as a function of individual firm cost) of any seller in a *symmetric* auction with N bidders. With costs uniform on [0, 1] and N players the well known equilibrium bid strategies are

$$P_{[N]}(c_i) = [(N-1)c_i + 1]/N.$$

To see this, momentarily dropping the subscript $[N]$, the profits of any bidder are

$$\pi_i(p) = \text{Prob}\left\{ p < \min_{j \neq i} P(c_j) \right\} [p - c_i],$$

where the notation $P(c_j)$ for the best response function of player j, rather than $P_j(c_j)$, exploits symmetry. P is strictly increasing in c, so we can define $c(p)$ as the inverse of $P(c_i)$. Then,

$$\begin{aligned} \pi_i(p) &= \text{Prob}\left\{ c(p) < \min_{j \neq i} c_j \right\} [p - c_i] \\ &= [1 - F_{[N-1]}(c(p))][p - c_i] \end{aligned},$$

where $F_{[N-1]}(.) = 1 - [1 - F(.)]^{N-1}$. For the uniform distribution over the unit interval, $F(c) = c$, so

$$\pi_i(p) = [1 - c(p)]^{N-1}[p - c_i].$$

The first order condition for profit maximization is

$$\begin{aligned} \frac{\partial \pi_i}{\partial p} &= -(N-1)(1-c(p))^{N-2}c'(p)[p - c_i] + [1 - c(p)]^{N-1} = 0 \\ &\Rightarrow -(N-1)c'(p)[p - c_i] + 1 - c(p) = 0. \end{aligned}$$

Here we have a differential equation. First we apply a boundary condition, *if* $c = 1$ then $p = 1$. That is, a firm will bid neither below its own costs nor above the maximal willingness to pay of the buyer. Second, we apply linearity which leads to the unique solution[19]

$$c(p) = (Np - 1)/(N - 1) \text{ or}$$
$$p = [(N - 1)c_i + 1]/N \equiv P_{[N]}(c_i).$$

While all results are dependent upon the realized cost draws, it is useful to study the expected equilibrium conditions. The first order statistic for N draws from costs distributed uniformly on [0, 1] is E[min $c_{\underline{i}}$] $= 1/(N+1)$. Substituting into the above solution, the expected winning price is $\bar{P} = 2/(N+1)$. Quite reasonably the monopoly price is 1, the maximal demand price, and the price as N goes to infinity is 0, the lowest possible cost given the normalization.

In Fig. 1, we illustrate the cases of $N = 3$ and $N = 2$, to provide some additional insight. Along the cost axis, one quarter is the expected value of the lowest cost in three cost draws (triopoly). Since the bid is linear in costs, the expected lowest cost (the first order statistic), one quarter, maps directly to the expected triopoly price, one half. One third is the duopoly expected lowest cost draw, and two thirds

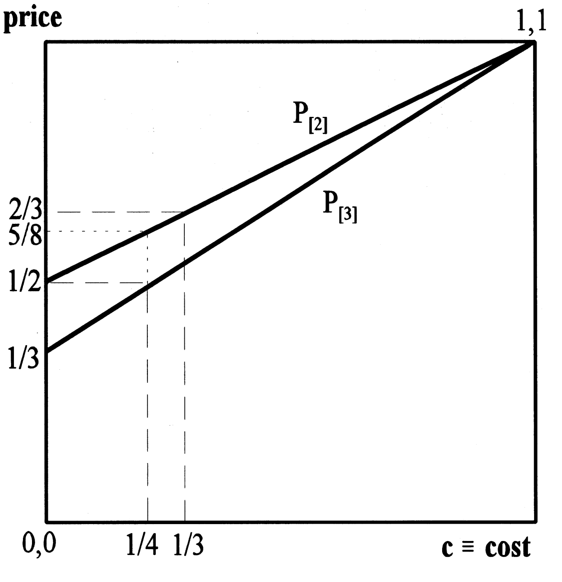

Fig. 1. Equilibrium price functions in symmetric duopoly and symmetric triopoly with uniform costs.

[19] This could be stated as a proof by (1) positing that the solution is linear, (2) solving the differential equation for the above values, and (3) verifying that the first order conditions solve to zero for all admissible c_i for these values. We show below that the solution is nonlinear in the asymmetric case.

is the corresponding expected value of the winning duopoly price. If one were to look at the price increase from moving from three to two firms in a $1/N$ symmetric model, one would overstate the expected price as two thirds, an increase of one sixth. This process implicitly assumes that the expected minimum cost increases from one quarter to one third, whereas a merger should not be assumed to throw away technologies leading to higher expected production costs. One might instead consider the effects of using the duopoly best responses with the *three* draw order statistic, projecting a postmerger price of five-eighths, an increase of one eighth. Although this does not ''throw away'' a technology, this too, vastly overstates the true price effect of a merger as firms ''act as if'' this technology were disposed of. To analyze the impact of the merger correctly, we next consider the asymmetric auction model.

3.2. The asymmetric model

In Appendix A we prove that an equilibrium within strictly increasing and differentiable strategies both exists and is unique.[20] The intuition underlying existence, the nature of the equilibrium, and hence the nature of the simulation, can be illustrated by looking at the simple case of $N=3$ cost draws, with $M=2$ bidders. We first need notation, for the ith firm, we define its number of cost draws as k_i. For our example, suppose that draws are distributed with $k_1=1$ and $k_2=2$. Then, defining c_i as the ith bidder's lowest cost draw (the technology it would use to fill an order),

$$c_1 \sim F_1^{(1)}(.) \equiv F(.) \text{ and}$$
$$c_2 \sim F_2^{(1)}(.) \equiv 1-[1-F(.)]^2$$

where the (*superscript*) on F denotes the first order statistic (expected lowest cost) for the cumulative distribution of firm i's costs. The probability density function of c_2 when $F(.)$ is the uniform distribution over the unit interval is given by $f(c)=2-2c$. Now the expected profit of bidder 2 is

$$\begin{aligned}\pi_2(p_2) &= \text{Prob}\{p_2 < P_1(c_1)\}[p_2 - c_2] = \text{Prob}\{c_1(p_2) < c_1\}[p_2 - c_2] \\ &= [1 - F(c_1(p_2))][p_2 - c_2] = [1 - c_1(p_2)][p_2 - c_2].\end{aligned}$$

Unlike in the symmetric case, it is important to use firm subscripts on the best response functions, e.g., $P_1(c_1)$ and its inverse applied to firm 2's price, $c_1(p_2)$.

[20] With discontinuous payoffs, a Nash equilibrium in pure strategies need not exist (Dasgupta and Maskin, 1986). To see the discontinuity suppose that, of two bidders, bidder 1 sets price p_1, where $p_1>c_2$. Then bidder 2 gets profits equal to (p_2-c_2) if $p_2<p_1$. Profits increase by increasing p_2 toward p_1 from below. But, at $p_2=p_1$ its profits are not equal to (p_1-c_2); there is a discontinuity in payoffs.

The final step above simplifies using $F(c)=c$, the uniform distribution over the unit interval, or $U[0, 1]$.

The expected profit of bidder 1 is

$$\pi_1(p_1) = \text{Prob}\{p_1 < P_2(c_2)\}[p_1 - c_1] = \text{Prob}\{c_2(p_1) < c_2\}[p_1 - c_1]$$
$$= [1 - F_2^{(1)}(c_2(p_1))][p_1 - c_1] = [1 - c_2(p_1)]^2[p_1 - c_1],$$

where $F_2^{(1)}$ is the two draw cumulative distribution function defined above with $F \equiv U[0, 1]$.

The first order necessary conditions simplify to

$$c_1'(p) = (1 - c_1(p))/(p - c_2(p))$$
$$c_2'(p) = (1 - c_2(p))/2(p - c_1(p))$$

with boundary conditions $c_1(a)=c_2(a)=0$ for some $a \in [0, 1)$ and $c_1(1)=c_2(1)=1$.

Assuming differentiability and exploiting the property that $c_i < P_i(c_i) < 1$ for all $c_i < 1$, we obtain price functions that are monotonically increasing.

The boundary conditions imply that for the best response functions and some point a, $P_1(0)=P_2(0)=a$ (the common vertical intercept), and that $P_1(1)=P_2(1)=1$ in the upper right corner of the unit square (as in Fig. 2 below). The second boundary condition implies that *if* $c=1$ then $p=1$. That is, a firm will bid neither below its own costs nor above the maximal willingness to pay of the buyer. The

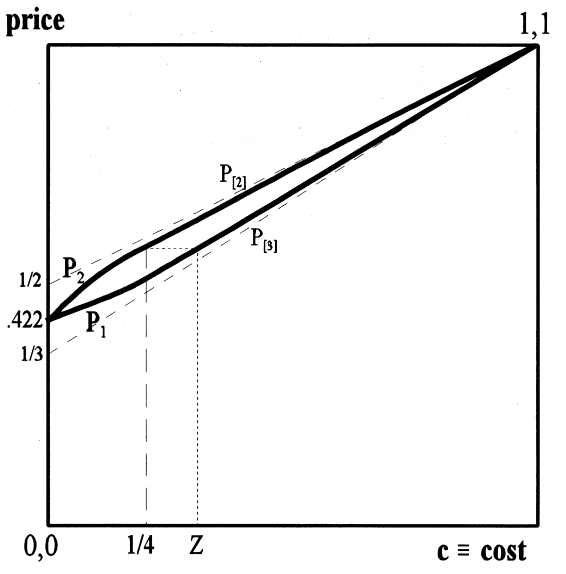

Fig. 2. Equilibrium price functions in asymmetric duopoly.

first boundary condition is also intuitive. Suppose that $P_i(0)<P_j(0)$. Then if firm i has costs $c_i=0$, it could raise its price slightly while maintaining certainty of being the winner. Hence $P_i(0)<P_j(0)$ cannot be a best response price for i, and $P_1(0)=P_2(0)\equiv a$ must be a boundary condition. Knowing the conditions for an equilibrium, if one exists, we state our main theorem informally:

Theorem 1: *The above problem has an equilibrium and the equilibrium is unique. A formal statement of the theorem and its proof are in Appendix A.*[21]

Given that a unique solution exists, numerical solution techniques can be used to calculate the expected winning price for any number of bidders with different numbers of cost draws. There is known to be a solution to this system of equations for some a, so one need only search across solutions satisfying $(p, c)=(a, 0)$ to find the unique one which also satisfies $(p, c)=(1, 1)$. The simulations were produced using a fourth-order Runge–Kutta method (Ixaru, 1984; Ascher et al., 1995), and graphical solutions were obtained via functional programming using *Mathematica*. For details, see Appendix B.

In Fig. 2, we illustrate the best response functions for the case outlined above: two of three firms (each with a single cost draw) merge and the merged firm (firm 2) has two cost draws. This is superimposed on the illustration of the symmetric single cost draw duopoly and triopoly results. Along the vertical axis, the intercept is $a=0.421875$. Along the cost axis, one quarter is the expected value of the lowest cost draw given three draws from the cost distribution. Unlike the symmetric case there is no graphical depiction of the expected value of the winning bid for two reasons: the best response functions are neither linear nor coincident.

Following our example, the expected triopoly price premerger is, as noted above, $\bar{P}_{[3]}=0.5$. In a duopoly in which firm 2 is the merged firm with two cost draws and firm 1 has one cost draw, the expected duopoly price is $\bar{P}_{\text{merger}}=0.573$.[22] These are prices based on the normalization of $c\in[0, 1]$; our calibration to more ''realistic'' cost distributions illustrates more modest price effects of merger.

[21] Lebrun (1996) provides a fairly general proof of existence in related auctions, and our auction may be a special case of those that he analyzes. Our proof, albeit a special case, is far more compact. Marshall et al. (1994) recognize the equilibrium existence difficulties and by simulation suggest that an equilibrium both exists and is unique, but do not prove this. They numerically calculate best response functions, stating that ''within numerical accuracy of . . . there is one and only one . . . '' solution to these first order conditions and boundaries.

[22] Dalkir (1995) shows that the expected price can be written in terms of the inverse equilibrium price functions. This expression is numerically integrated across the price range to estimate the expected price.

Having characterized the solution for an example, we turn to some efficiency results.

3.3. Contrasting asymmetric and symmetric auction efficiency

A first-price auction mechanism with symmetric independent private values exhibits the *revenue equivalence* property. It generates the same expected revenue and allocation as a second-price auction mechanism, the highest-revenue direct-revelation auction for the independent private-values model. The bidder with the highest valuation (lowest cost) receives the good (contract).[23] Revenue equivalence does not hold for asymmetric auctions (Maskin and Riley, 1996). Simply put, with symmetric auctions all first price auction response functions are identical so that the firm with the lowest cost draw bids the lowest price. But, in the asymmetric auction, the first price auction firms' reaction functions are not coincident. The good might not go to the bidder with the lowest cost in the first price auction (whereas it would go to the lowest cost bidder in the second price auction).

This implies a cost inefficiency may arise in the asymmetric first price auction. Using Fig. 2 above, if $c_2 = 0.25$ and $Z > c_1 > 0.25$, firm 1, despite higher costs, wins the auction. The first impression is that mergers lower efficiency in auction markets. But this is solely because we suppose that the merger creates asymmetry. One might consider an initially asymmetric auction in which a merger creates a symmetric auction and the potential for cost-inefficient bid winning is eradicated by the merger.

3.4. An asymmetric auction with a merger to symmetry

Since we have characterized the asymmetric auction, we look at the solution to a symmetric auction created by the merger of asymmetric firms. In our context, this involves a symmetric auction with each firm having $k \geqslant 2$ cost draws.

Suppose that post merger $R \geqslant 3$ firms each have k cost draws: $Rk = N$, where N is the total number of cost draws. The best response functions for a game with R firms each with k cost draws are identical to the best response functions for the game with $N - k + 1$ firms, each with a single cost draw. (A proof appears in Appendix C.) The intuition for this result is straightforward. In the former game each bidder faces $R - 1$ rivals with $(R - 1)k$ cost draws. The first order statistic on $(R - 1)k$ cost draws determines the expected lowest cost held by a rival. In the latter game each bidder faces $N - k$ rivals each with a single cost draw. The expected lowest cost of a rival comes from the first order statistic from $N - k$ cost draws. Then note that $(R - 1)k = N - k$. The rest of the intuition is immediate. Suppose that the best response rule followed by one firm's *rivals* in the former game is identical to that in the latter game. Then clearly the first order condition

[23] See, for example, McAfee and McMillan (1987).

for this firm in the former game will be equal to that in the latter game (same marginal probability of losing as price increases) so it will select the same best response rule as well. From here it is simple to obtain the equilibrium expected price, it being the mapping of the first order statistic from N cost draws into the best response function for the $N-k+1$ single cost draw best response function.

To illustrate briefly, consider a triopoly with two firms each with a single cost draw and one firm with two draws. Merging the first two firms leads to a symmetric duopoly; each firm has two draws. For the uniform $c\in[0, 1]$ the asymmetric premerger price is 0.430, and post merger it is 0.467.

We turn now to calibrated models simulating potential biases in merger analysis.

4. Application to merger analysis

There are three issues we wish to address: (1) the profitability of merger, (2) the calibration of the model and ''realistic'' price increases from mergers and (3) the sensitivity of the model to our distributional assumptions. We deal with these in order.

4.1. Is merger profitable?

One question is whether the mergers we model would be profitable to undertake. For policy modeling of mergers one should not use a model which predicts mergers which would not be privately chosen as optimal by the firms involved (cf., Salant et al., 1983).

Modeling the merger incentive prior to the cost draw, the answer to this question is ''Yes,'' as all of our simulations demonstrated. While proof would require several steps, the logic can be expressed simply by looking at one case. Firm i supposes the following. ''If I am the lowest cost firm I will win the auction at a profit, and if I am not I will earn zero'' (which is correct in the symmetric premerger case). Now, suppose that i merges with some other firm (which changes the possession of the cost draws but not their distribution). The expected value of the merged entity would be exactly double the expected value of the unmerged entity if nothing changes (i's best response or its rivals'). But, if it merges, it realizes that with some positive probability it has eliminated the second lowest cost firm, leading to a higher best response price as a function of the lower of its two cost draws. By definition of a best response, firm expected profitability of the merged entity must hence exceed the profitability of the sum of the two entities if there were no merger. Hence i's (with a partner) expected price rises. This in turn means that the $M-2$ firms with a single cost draw each will find it profitable to raise their best responses above the M firm symmetric best responses. This only makes it more profitable for firm i (and its partner). Next, suppose that for some rivals' (symmetric/identical) best response rule the merged entity were to have lower profits than the sum of the premerger profits of the two entities. Then the merged entity could select instead to bid the premerger best response rule for its lowest cost draw and its expected profits would be greater if its rivals had higher best response rules, and no lower if they had their initial best response rules. If this were to lead to higher expected profits then this must dominate a best response price which exceeds this, a contradiction. The best response would then be the initial best response (or lower).

We now only need to eliminate this counterfactual, that prices would instead be lower. Suppose the ith firm were to bid its initial best response rule. Its profits would be lower only if its rivals were to bid *lower* prices due to the merger. For the rivals to bid lower prices their expectations must be that the merger would lead to a lower price best response rule for the lower cost draw of the two draws for the merged firm i. Firm i would never have an incentive to bid a lower price unless its rivals *independently* had an incentive to do so, and in our context firm i could never ''credibly commit'' to do so. Hence it is clear that mergers are profitable.

In contrast, Marshall et al. (1994) find that while coalitions are sustainable and profitable for small M, they are not sustainable for large M. With $M=101$, and a coalition of 100 equally sharing coalition profits, they demonstrate that the 101st firm would not wish to join the coalition. With a coalition of 101 firms, one firm could leave the coalition and earn greater profits. In their coalitions, all firms equally share in the profits. For merger analysis, however, an acquiring firm can choose how much profits to offer to a potential merger partner, which can in turn decide whether to accept the merger proposal. For a merger one can calculate from their table that for one firm with 100 cost draws and one firm with only one cost draw a Pareto-preferable merger exists. Following a merger logic, the 100 draws are not separate firms in a coalition capable of withdrawing and free-riding. Bribing a firm to join, i.e., paying a purchase price in excess of opportunity costs, leads to no free-rider problem as that final ''plant'' or ''draw'' becomes part of a single legal entity.[24]

4.2. Calibration and simulation of merger price effects

Calibration of such a model depends upon the available data. When possible, econometric analysis of the market can yield estimates of marginal costs, demand

[24] We have not dealt with the ''hold out'' problem. That is, we do not prove that there exists a series of mergers that would lead 101 firms to merge sequentially, only that at the margin one merger is profitable. Waehrer (1997), for example, demonstrates that one would prefer to free ride others' mergers in such models.

elasticities and the like.[25] Here, we proceed by calibrating the model via market shares and the coefficient of spread (spread/mean ratio).[26]

There are two elements to calibrate: (1) market shares prior to merger and (2) the range of the density function of costs, $[\mu - \Delta, \mu + \Delta]$ or $[\mu(1-d), \mu(1+d)]$ where $d \equiv \Delta/\mu$ is the coefficient of spread. For an example, start with a symmetric market. Then, normalizing cost draws from the range $c \in [0, 1]$ to $c \in [100(1-d), 100(1+d)]$, we can calibrate the model to the coefficient of spread for a single bidder. In actual merger cases reasonable estimates of d can be elicited from the technical staff (''engineers'') or from the administrators who are informed about costs. The premerger expected winner's price is $\bar{P} = 100[4d/(N+1) + 1 - d]$ and the premerger expected winner's cost is $\bar{c} = 100[2d/(N+1) + 1 - d]$.

Suppose that d is between 5% and 25%, in the real world d is unlikely to exceed 25%. Then one can derive the normalized premerger price and cost for a given N to simulate the effects of merger. We illustrate with mergers of symmetric firms, in markets with $N=3$, $N=4$, and $N=6$, in Table 1 below. Similarly, it is possible to calibrate the price increase from asymmetric firms merging to become symmetric. To demonstrate, consider the merger of three asymmetric firms ($M=3$) into two symmetric firms, as in the above section. The results are shown in Table 1.

Despite the small numbers of firms, the simulations suggest that the unilateral effects from a merger (using the true model) are modest at worst. For example, the Guidelines often focus on a five percent price increase as ''significant.'' None of the simulated price increases is greater than five percent. This may be a convincing basis for approval of a merger, especially if moderate efficiency gains are likely (cf., Willig, 1997).

From Table 1, the percent increase predicted by the ''naive'' $1/N$ formula is slightly higher than *M times* (resp. *N times*) the true increase for an asymmetric (resp. symmetric) merger. This provides a rough rule of thumb for inferring the true price increase from the $1/N$ formula. We also mentioned that we could bound price effects by looking at price increases from mergers using the $1/N$ and $1/(N-1)$ best response functions, but using the limit statistic for N cost draws. This is used in the final column in Table 1. This roughly eliminates half of the bias from using the $1/N$ rule, but with such a huge bias from the $1/N$ rule, the biases

[25] In individual markets these bids are few and contracts are of long duration. Hence standard time series cannot be of much use. To use cross sections to calibrate one must assume that all of the markets have the same technologies and demands. Furthermore, if one has the cross section information to calibrate this model and one is willing to assume that each market has the same technological and demand conditions, then one should have sufficient data to predict the effects of market structure on price directly through regression simulation without imposing this theoretical structure.

[26] Alternatively, we could have chosen to calibrate using the coefficient of variation, σ/μ, which, for the uniform distribution, is equal to $d/\sqrt{3}$. Calibration could also start from premerger Price Cost Margins or other data.

Table 1
Simulated price effects of mergers

Market structure: Premerger	Market structure: Postmerger	d (percent)	Initial expected price	Initial expected cost	Expected price increase	Percent price increase	Percent increase: $1/N$ model	Percent increase: Bounded model
Symmetric	Asymmetric	5.00	100.00	97.50	0.73	0.73	1.67	1.25
$N=3$	$M=2$	16.67	100.00	91.67	2.42	2.42	5.56	4.17
		25.00	100.00	87.50	3.64	3.64	8.33	6.25
Symmetric	Asymmetric	5.00	99.00	97.00	0.30	0.31	1.01	0.67
$N=4$	$M=3$	16.67	96.67	90.00	1.01	1.04	3.45	2.30
		25.00	95.00	85.00	1.51	1.59	5.26	3.51
Symmetric	Asymmetric	5.00	97.86	96.43	0.09	0.09	0.49	0.29
$N=6$	$M=5$	16.67	92.86	88.10	0.30	0.33	1.71	1.30
		25.00	89.29	82.14	0.45	0.51	2.67	1.60
Asymmetric	Symmetric	5.00	99.30	97.00	0.36	0.37	1.67	1.25
$M=3$	$R=2$	16.67	97.67	90.00	1.22	1.24	5.56	4.17
($N=4$)		25.00	96.51	85.00	1.82	1.89	8.33	6.25

are still substantial. The bound in, for example, the $M=5$ case is over three times the asymmetric model actual calculations.

The first and last cases in Table 1 are both models in which a triopoly merges to become a duopoly. The first is a merger from symmetry to asymmetry, the latter is from asymmetry to symmetry. This suggests that the percent price increase from merger to symmetry is about half of the percent price increase from merger to asymmetry, implying that price effects of mergers leading to more symmetry are much smaller than those creating more asymmetry.

4.3. Uniform vs. Nonuniform distributions

One calibration issue is whether it is sensitive to the chosen density function, e.g., the uniform. Without a detailed analysis here we examine sensitivity with an example. Suppose that $N=6$ and that each of three firms has two cost draws from a uniform distribution (thus the minimum cost draw is not from the uniform). Again, a merger of two firms would lead to an asymmetric duopoly, one firm with two draws another with four. In the column labeled ''Minimum of k Uniforms'' in Table 2 we contrast the calibrated results for three symmetric firms and their merger to an asymmetric duopoly using both the initial condition above, $N=3$, the first three rows, and the new model for which $N=6$, the last three rows.

The observation of the ''Minimum of k Uniforms'' percent price increases for the two asymmetric cases leads to two related practical implications. First, because of calibration to some known market phenomenon, the restrictive form of the density function is not likely to lead to significant errors for ''estimation.'' Second, close models will have close outcomes, a desirable property for simulation of asymmetric initial conditions. Elaborating on the second point, consider a market

Table 2
Calibrated price increases: uniform cost distribution vs. nonuniform cost distributions

Market structure		Percent spread d of the uniform	Percent coefficient of variation	Percent price increase	
Premerger	Postmerger			Minimum of k uniforms	Extreme value
Symmetric	Asymmetric	5.00	2.89	0.73	0.54
Triopoly	Duopoly	16.67	9.62	2.42	1.83
$k=1$	$k_{\text{merger}}=2$	25.00	14.43	3.64	2.76
	$k_{\text{nonmerger}}=1$				
Symmetric	Asymmetric	5.00	2.40	0.66	0.54
Triopoly	Duopoly	16.67	8.32	2.31	1.92
$k=2$	$k_{\text{merger}}=4$	25.00	12.86	3.58	3.01
	$k_{\text{nonmerger}}=2$				

with three firms with asymmetric market shares. One could simulate shares using the uniform distribution on [0, 1] by varying cost draws, $k_1+k_2+k_3=N$. If one had a similar market in terms of shares, one might require a significantly different N (e.g., $9/32$ is very close to $1/4$) to simulate the initial shares. The stability of the *calibrated* results (not those on [0, 1]) with respect to N implies that for our algorithm similar starting shares yield similar results on merger consequences.

Our distributional assumption is likely closer to being an upper bound on the extent of the merger price increase, over different types of distributions. For further sensitivity analysis with respect to the type of the distribution, we compared our results with those from an extreme value distribution,[27] as in Tschantz et al. (1997).[28] In the above section we used the spread parameter d to calibrate the price increases under the uniform; we do the same in Table 2. In order to hold the coefficient of variation constant across the two distributions, we need to calibrate the extreme value distribution to the coefficient of variation implied by d. Each value of the spread parameter d implies a different value for the standard deviation σ, and hence a different coefficient of variation, σ/μ. The coefficient of variation (σ/μ) of a uniform distribution can be expressed as a simple function of its spread parameter d, as $\sigma/\mu=d/\sqrt{3}$.

On the first set of rows of Table 2 (premerger $k=1$), the expected price increases under "the minimum of k uniforms" are equivalent to those on the first row of Table 1 under the true model with uniform costs (minimum of one uniform random variable is equivalent to the uniform random variable itself). On the same row, the price increases under "extreme value" are coming from an extreme value distribution with the same coefficient of variation σ/μ corresponding to each d.

[27] The extreme value density is $f(x)=\exp[-(x-a)/b]\ \exp\{-\exp[-(x-a)/b]\}/b$ for $-\infty<x<+\infty$ with mean$=a+(0.57721)b$ and variance$=b^2\pi^2/6$.

[28] We would like to thank Luke Froeb for assistance in this calibration.

On the second set of rows of Table 2 (premerger $k=2$), the expected price increases under ''the minimum of k uniforms'' are from a merger between two out of three symmetric firms, each with two cost draws; each cost draw is distributed uniformly with $\mu=100$ and spread parameter d. Now the minimum of two uniform costs does not have a uniform distribution, and its mean and variance are different than those of the underlying uniform distribution. The coefficient of variation (σ^*/μ^*) of the *minimum* cost is related to the spread parameter d of each of the two *uniform* costs, as $\sigma^*/\mu^*=2d\mu/(3\sqrt{2}\mu^*)$. The expected price increases under ''extreme value'' are coming from an extreme value distribution with a coefficient of variation σ^*/μ^*.

We display the percent price increases for the two types of distributions in Table 2. For the asymmetric merger case with uniform costs (premerger $k=1$), the percent price effect from the extreme value distribution is approximately three-quarters of our prediction. For the nonuniform case (premerger $k=2$), it is approximately 83% of ours.

Some variation is shown to alternative choices of the distribution used in the calibration, but the results changed little. And, for whatever variation does exist across distributions, it appears that the uniform, with its ''fat tails'' tends to represent a ''worst case'' scenario in terms of merger price effects.

5. Compensating merger efficiencies

Welfare effects in auction markets are usually small because of the perfectly inelastic demand; the only welfare loss occurs in a first price auction when the ''wrong'' firm wins the auction, not because the buyers turn away from the product. Enforcement agencies may ask how much the expected cost of the merging parties would have to go down (i.e. the size of the ''merger-specific efficiencies'') in order to counteract the price effect of the merger.

To get an idea about the magnitude of the merger-specific cost savings that would exactly offset the merger's price effect, we first calibrated the merger firm's cost distribution such that the postmerger price is the same as the premerger price in the absence of efficiencies. Technically, we redefined the merger firm's cost distribution as $F_2^{(1)}(.)=1-[1-F(.)]^{2+\epsilon}$ where $F(c)=c$ is the uniform distribution, while keeping the nonmerger firm's distribution as the uniform. This has the effect of lowering the mean and the variance of the distribution, while holding its support.[29] We increased ε until the postmerger equilibrium price (from new best response price functions for both firms based on the new set of distributions) was

[29] The distribution $F^{(1:z)}(.)=1-[1-F(.)]^z$ has the nice interpretation of being the distribution of the minimum of z independent draws from distribution F, where z is an integer. With merger-specific efficiencies, we can model this as if the firm had an additional (fractional) draw from the distribution. The mean of $F^{(1:z)}$ when F is the uniform over [0, 1] is $\mu^{(1:z)}=1/(z+1)$. Accordingly, if prior to merger we have $z=1$ and post merger we treat this as $z=2$, we can model efficiencies by simply using the fractional form, $z=2+\varepsilon$ which has a lower mean cost.

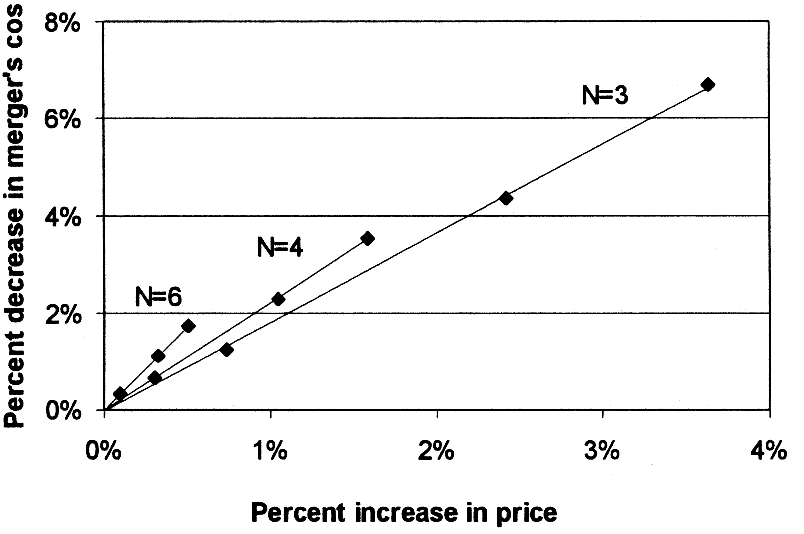

Fig. 3. Price offsetting efficiencies.

equal to the premerger price, at some ε^*. We then estimated the merger-specific efficiency gain that exactly offsets the merger's price effect as the difference between the expected cost from the cost distribution with $\varepsilon = \varepsilon^*$ and the expected cost from the cost distribution with $\varepsilon = 0$ (no merger-specific efficiencies).

Fig. 3 is a scatter plot of the merger-specific efficiencies needed to offset a given change in the market price. The three curves displayed in the figure, with approximate slopes[30] of 3.5, 2.2 and 1.8, correspond to the merger of two firms out of six, four, and three firms, respectively. On each curve, the three dots correspond to the calibration of the spread parameter d to values of 5% (closest to the origin), 16.67%, and 25% (farthest from the origin).

For example, in the merger of two out of three symmetric firms with $d = 25\%$, a 6.70% reduction in the expected cost is needed to offset the 3.64% price increase. In the merger of five out of six symmetric firms with $d = 5\%$, a 0.32% reduction in the expected cost is sufficient to offset the 0.09% price increase. These imply that in the mergers of symmetric firms, merger-specific efficiencies of approximately 1.8–3.5% would be sufficient to offset a 1% price increase.[31]

[30] The true curves are not straight, their slopes increase slightly with the percentage change in price.

[31] Cf. Brannman and Froeb (1997), who estimate that a 4% (or higher) cost reduction is necessary to offset a 1% price increase from a merger in a second-price auction model. Tschantz et al. (1997) estimate that approximately a 2.5% cost reduction is necessary to offset a 1% price increase from a "small" merger in a first-price auction model, where the merger's share is less than 69%. Both of these models use the extreme value distribution. The figure for the first price auction is well within the range of our estimates. The figure for the second price auction is higher because in a second price auction, merger-specific efficiencies affect the equilibrium price only when the merger firm "just" loses an auction, not when it wins.

Williamson (1968) showed that welfare gains from merger efficiencies can offset the losses due to a higher price. We assume that a merger's expected cost is lower than either of the merging parties' premerger expected cost; simultaneously there is an increase in the expected price due to the change in the number of firms. Consequently, the merger has to aim for additional efficiencies if it wants to balance off the price effect. Our results confirm the basic Williamsonian insight in a specific equilibrium oligopoly equilibrium model.

6. Extensions

Market-specific issues may arise in a particular application. To explore these, we return to the application to hospital mergers. Hospitals may provide a vector of services, e.g., services A and B (cardiac and obstetrics). In the context of the above model this leads to another implication for mergers (cf., Dalkir, 1995 for a formal treatment). Suppose that for a PPO, a hospital must bid for the vector of hospital services (which is what we observe in practice). Each hospital receives a cost draw for each service, A and B.[32] A merged hospital selects the lower cost of two signals for procedure A and for procedure B. It is then simple to demonstrate that for PPO bids, a merged hospital has a lower expected cost for the combined sale of A and B than the expected costs of the two nonmerged entities. Hence, hospital mergers are likely to yield cost savings since only one bid winner provides the complete vector of services to the PPO.

In an application Dalkir (1995) demonstrates that for i.i.d. cost draws, initial calibrations, and some market structures, this cost advantage from the combination of services may be substantial. Merger price increases from such a model will be less than in the corresponding single service model and the efficiency gains may even outweigh price effects. This could not be demonstrated using either the naive or bounded models. In particular for such market structures the importance of using the asymmetric auction model for merger simulation establishes a presumption for a merger that could not be established by use of an analytically simpler symmetric auction model.

7. Concluding remarks

Despite the new trend towards use of unilateral effects models for merger analysis, still most merger cases, even ones for which the presumption is one of

[32] For the effects noted, one must assume that the cost draws are not linearly dependent, e.g., the cost of A is not α times the cost of B for both hospitals.

unilateral action, are decided using ''structural indices.'' That is, concentration (HHI) levels and changes play an important role, whereas modeling does not play a big role in most cases. As this is shifting to more use of explicit unilateral effects modeling, our modeling approach may be useful for calibration in more cases when they involve auction sales. As we demonstrate, the analytically tractable $1/N$ models would lead to misleadingly huge calibrated unilateral effects of mergers in auction markets. Our method of solution for the asymmetries implied by mergers show that these are important to understand when predicting price effects of mergers and that, when unilateral effects, rather than collusion, are the concerns of the antitrust agencies, that price effects in auction mergers appear to be quite modest relative to what one would expect from application of these simpler models.

Having a model is also useful for other considerations. For example, the prospects for entry post merger will change with a merger. Indeed, we can see this by looking at the first two cases in Table 1 (using $d=16.67$ for simplicity). If there are $N=3$ symmetric firms, the initial expected price is 100, entry of one more (identical) firm would lead to four firms and a price of 96.67. If there were a merger from $N=3$ to $M=2$, and no efficiency effects, then price would be 102.42, and entry would lead to the case labeled $M=3$ below, with expected price of 97.68, entry is more likely post merger. With predictions like these and knowledge of industry characteristics, profitability, potential entrants and the like, antitrust agencies can assess with greater accuracy the likelihood of entry in the absence of efficiency gains from the merger.

This type of ''entry'' information is particularly important to know for auction markets in which bid preparation is costly. In some cases the cost of bid preparation substantial. (E.g., it may involve costly exploration or significant R and D simply to respond to a request for a proposal.) In such cases there may be more producers of the product than there will be bidders in a two stage game in which firms must first ''prepare the bid'' and then there is ''bidding by those firms which prepared a bid.'' But, it may be that there is very little change in profitability of entry (preparing the bid) before an additional firm would enter into the auction. These and other factors can be used to increase the power of the model in actual application.

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Appendix A

Proof of the main theorem[33]

The asymmetric first-order equilibrium conditions derived in the text assume that the equilibrium strategies are strictly increasing, hence invertible.[34] Additionally, the proof rests upon the following two arguments: (i) that the set of possible solutions for the asymmetric first-order conditions is connected, and (ii) that the strategies of the two asymmetric players do not cross in the interior of the price$\times$cost space.

The equilibrium requires solving the differential-equation first order conditions with two boundary conditions, a common initial point, a, where $c=0$ and a common terminal point of value 1 where $c=1$. Although one is assured of a solution to a system like our first order conditions with one boundary condition (cf., Theorem 11.1 in Ross, 1964), there is no general proof for the existence, or uniqueness, for the solution of two differential equations with two boundary conditions. Using standard differential equations notation, we provide a proof herein. In this notation, $y(t)$ is our inverse best response function ($c(p)$ in the text), and t_0 is the initial condition (a in the text).

With this notation, our first order conditions are the inverse price functions

$$y_1' = (1 - y_1)/(t - y_2),\ y_2' = (1 - y_2)/2(t - y_1)$$

which satisfy for an initial value $t_0 \in (0, 1]$, the *initial condition* and $t=1$, the *terminal condition*. Denote, using vector notation, $y \equiv (y_1, y_2)$, $f(t, y) \equiv (f_1(t, y), f_2(t, y))$, where $f_1(t, y) \equiv (1-y_1)/(t-y_2)$ and $f_2(t, y) \equiv (1-y_2)/2(t-y_1)$. We then have the Boundary Value Problem

$$\text{(BVP)} \begin{cases} y' = f(t, y) \\ y(t_0) = (0, 0) \\ y(1) = (1, 1) \end{cases}$$

Then ask whether there is any $t_* \in (0, 1]$ such that when $t_0 = t_*$, the solution of the Initial Value Problem

$$\text{(IVP)} \begin{cases} y' = f(t, y) \\ y(t_0) = (0, 0) \end{cases}$$

[33] We thank Kevin Hockett of The George Washington University, Jerrold E. Marsden of Caltech and Lars Wahlbin of Cornell University for their comments on this proof.

[34] We can justify a priori that the equilibrium strategies must be nondecreasing, but strict monotonicity is verified by the actual shape of the numerically-estimated equilibrium strategies.

is also a solution of (BVP). Let $y^{t_0} = (y_1^{t_0}, y_2^{t_0})$ be the notation for the pair of functions that solve (IVP). Formally our goal is to prove:

Theorem. *There is a unique* $t_* \in (0, 1]$ *such that the solution* $y^{t}*$ *of* (*IVP*) *satisfying* $y^{t}*(t_*) = (0,0)$ *is also a solution of* (*BVP*), *satisfying* $y^{t}*(1) = (1, 1)$.

After some definitions and lemmas we prove this Theorem. The method of proof is as follows. First we separate all possible solutions of (IVP) over the unit square[35] into two subsets, and call these ''terminated'' and ''continued'' solutions. In Fig. A.1 below, the two solutions on the left are terminated and the two on the right are continued. We also show that the two components y_1 and y_2 of any given solution $y = (y_1, y_2)$ never cross inside the unit square (*Remark 2* below). We then invoke a known theorem to show that the set of all solutions is connected, and therefore there must be a solution that is adjoint both to the set of terminated solutions and to the set of continued solutions. Finally, we show that the two components of that ''connecting'' solution approach each other as we move to the right on the horizontal axis of Fig. A.1, and they meet exactly at the upper right corner of the unit square (*Lemmas 6* and *7*); we also show that the ''connecting'' solution is unique. This ''connecting'' solution of (IVP) is also the unique solution of (BVP).

Definition 1. A solution of (IVP) is a continuous function y^{t_0} defined over $[t_0, \tilde{t}_0)$ where $\tilde{t}_0$ is called the termination point of y^{t_0}, and is defined below.

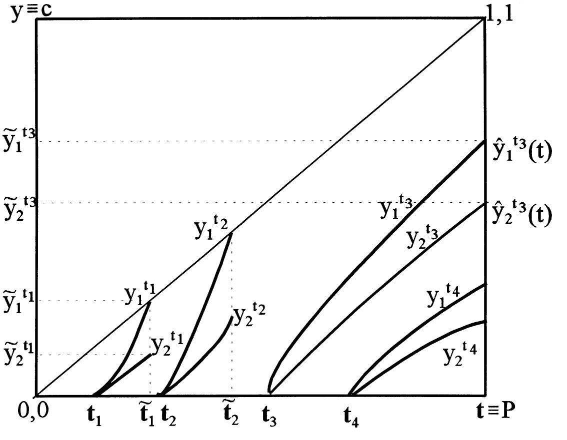

Fig. A.1.

[35] To be precise, the unit cube. However, we will continue to refer to the unit square mainly for graphical convenience.

Remark 1. (*IVP*) *has a unique solution* y^{t_0} *for any* $t_0 \in (0, 1]$ *on any interval over which f satisfies the Lipschitz condition, by "The Basic Existence Theorem" for ODEs* (Ross, 1964). *This theorem holds for* (*IVP*) *because f is differentiable, and differentiability implies that the Lipschitz condition is satisfied.*

Definition 2. The termination point, or limit fixed point, if any, of y^{t_0} such that $y_1^{t_0}(t) \to \tilde{t}_0$ as $t \to \tilde{t}_0$.

Definition 3. If $y_1^{t_0}(t)$ has a termination point $\tilde{t}_0 < 1$ then y^{t_0} is called a terminated solution defined over $[t_0, \tilde{t}_0)$. If $y_1^{t_0}(t)$ does not have a termination point $\tilde{t}_0 < 1$ then y^{t_0} is called a continued solution defined over $[t_0, 1]$. If y^{t_0} is a continued solution, then we let $\tilde{t}_0 = 1$ for notational convenience.

Definition 4. An extension of a terminated solution y^{t_0} is

$$\hat{y}^{t_0}(t) = y^{t_0}(t) \text{ over } [t_0, \tilde{t}_0),$$

$$\hat{y}^{t_0}(t) = \tilde{y}^{t_0} \text{ at } \tilde{t}_0,$$

where $\tilde{y}^{t_0}$ is defined as $\lim_{t \to \tilde{t}} y^{t_0}(t)$. We also define $(\hat{y}^{t_0\prime}(\tilde{t}_0)$ as $\lim_{t \to \tilde{t}_0} (y^{t_0})'(t)$, which needs not be finite.

Lemma 1. *There exists some separation point* $s \in (0, 1]$ *such that for* $t_0 < s, y^{t_0}$ *is a terminated solution of* (*IVP*) *and for* $t_0 > s$, y^{t_0} *is a continued solution of* (*IVP*).

Proof. It is easy to show that the second derivatives can be written as

$$y_1'' = y_1'(y_2' - 2)/(t - y_2) \text{ and } y_2'' = y_2'(y_1' - 3/2)/(t - y_1).$$

One can then write

$$\{y_1'' > 0 \Leftrightarrow y_2' > 2\}, \{y_2'' > 0 \Leftrightarrow y_1' > 3/2\},$$

since $y_1' > 0$ and $y_2' > 0$. Given those, and $y_1' = (1 - y_1)/(t - y_2)$ and $y_2' = (1 - y_2)/2(t - y_1)$, we claim that for all $t_0 < \frac{1}{4}$, y^{t_0} is terminated and for all $t_0 > \frac{2}{3}$, y^{t_0} is continued. Therefore we claim that both types of solutions actually exist.

For all $t_0 < \frac{1}{4}$, y^{t_0} is terminated because not only are y_1 and y_2 "steep" (their slopes exceed 4 and 2 respectively at t_0), but also convex (y_1 is convex because the slope of y_2 is greater than 2, and y_2 is convex because the slope of y_1 is greater than 3/2), therefore they become even steeper (reinforcing each other's convexity) for greater values of t; y_1 intercepts the diagonal at some $\tilde{t}_0 < 1$. Similarly, for all $t_0 > \frac{2}{3}$, y^{t_0} is continued because not only are y_1 and y_2 "flat" (their slopes are smaller than 3/2 and 3/4 respectively at t_0), but also concave (y_1 is concave because the slope of y_2 is less than 2, and y_2 is concave because the slope of y_1 is less than 3/2), therefore they become even flatter (reinforcing each other's concavity) for greater values of t, and remain below the diagonal for all values of $t \leqslant 1$. We conclude that both types of solutions exist.

Now suppose that *Lemma 1* is not correct, then there must exist initial points t_1, t_2, t_3, t_4 such that y^{t_1}, y^{t_2} are terminated and y^{t_3}, y^{t_4} are continued (see Fig. A.1), and such that one can find t_5 in the open interval (t_1, t_2) with y^{t_5} continued, or t_6 in the open interval (t_3, t_4) with y^{t_0} terminated, or both. This contradicts the uniqueness of the solution to (IVP) at some point. From Remark 1 above, uniqueness in (IVP) implies that solutions starting from different initial points cannot cross, but this is exactly what would happen if the set of solutions was not separated. QED

Definition 5. Let $S(I)$ define the set of all solutions of (IVP) defined over interval $I \subset (0, 1]$ and let S^1 be the notation for the set of solutions of (IVP) with initial points in the interval $J \subset (0, 1]$. Then define

$$S_{\mathrm{T}}(I) = \{y \in S(I),\ y \text{ is terminated}\}$$
$$S_{\mathrm{C}}(I) = \{y \in S(I),\ y \text{ is continued}\}.$$

S_{T}^J and S_{C}^J are defined for $y \in S^J$ similarly.

Remark 2. *The strategies of the two asymmetric players do not cross in the interior of the price $\times$ cost space, that is, $y_1(t)$ and $y_2(t)$ do not cross at any $t<1$ for any solution $y(t)$ of (IVP). Indeed, observe that $y_1 > y_2$ in some neighborhood of t_0 because $y_1'(t_0) > y_2'(t_0)$. Then let $t_1 > t_0$ be the first point at which y_1 and y_2 cross (but are not equal to one, which is what we are trying to prove). To this end, let $y_1(t_1) = y_2(t_1) = \alpha < 1$ for some $\alpha > 0$. Then $y_1'(t_1) = (1-\alpha)/(1-\alpha) = 1$ and $y_2'(t_1) = (1-\alpha)/2(1-\alpha) = 1/2$, i.e., at t_1, y_1 is steeper than y_2, a contradiction. Since $y_1 > y_2$ at all points to the left of t_1, y_2 has to be steeper than y_1 at t_1.*

Remark 3. *It follows that $\tilde{y}_2 \neq \tilde{y}_1$ implies $\tilde{y}_2 < \tilde{y}_1$ because $y_2 \leqslant y_1$ for all y over $[t_0, \tilde{t}_0)$ (from Remark 2).*

Lemma 2. *Let $\{t_n\}$ be a sequence in the open interval $(0, 1)$, with limit t_0. Over an interval on which the solutions of (IVP) y^{t_n}, y^{t_0} exist, $y^{t_n} \to y^{t_0}$ uniformly as $t_n \to t_0$.*

Proof. Follows from the continuous dependence of the solutions with respect to the initial values as stated and proven by, for example, Pontryagin (Pontryagin, 1962, Theorem 15, pp. 179–181).

Lemma 3. *If $y^{t_n} \to y^{t_0}$ on $[t_0, \tilde{t}_0)$ then $\hat{y}^{t_n} \to \hat{y}^{t_0}$ and $(\hat{y}^{t_0})' \to (\hat{y}^{t_0})'$* on $[t_0, \tilde{t}_0]$.

Proof. Follows from *Lemma 2* and the continuity of $\hat{y}$ and $\hat{y}'$.

Lemma 4. *$S^J(I)$ is connected for any I. Moreover, if J is compact then $S^J(I)$ is compact.*

Proof. Let Γ be the mapping, $\Gamma: J \to S^J(I)$ for arbitrary subintervals I, J of (0, 1] such that $\Gamma(t_n) = y^{t_n}$ on I. From *Lemma 2*, Γ is continuous. Since J is connected, $S^J(I)$ is connected. If J is compact, so is $S^J(I)$. QED

Lemma 5. *Let $\hat{S}(I)$ be the set of solutions over I extended from $[t_0, \tilde{t}_0)$ to $[t_0, \tilde{t}_0]$ by taking limits. Then $\hat{S}(I)$ is connected.*

Proof. Follows from *Lemmas 3* and *4*.

Lemma 6. *There is no solution such that $\tilde{y}_2 < \tilde{y}_1 = 1$.*

Proof. Suppose $y \in S$ satisfies $\tilde{y}_2 < \tilde{y}_1 = 1$. As above, we can write

$$y_1'' = y_1'(y_2' - 2)/(t - y_2) \text{ and } y_2'' = y_2'(y_1' - 3/2)/(t - y_1),$$

hence

$$\{y_1'' > 0 \Leftrightarrow y_2' > 2\}, \{y_2'' > 0 \Leftrightarrow y_1' > 3/2\}.$$

Moreover, it can be shown that

$$y_2''' = \frac{(y_2'')^2}{y_2'} + \frac{y_2' y_1'' + y_2''(y_1' - 1)}{t - y_1}.$$

If $y_2 < 1$ and $y_1 \to 1$ as $t \to 1$ then $y_2' = (1 - y_2)/2(t - y_1) \to +\infty$. Now we will first prove that $y_2'' > 0$ as $t \to 1$, and then show that there is a contradiction.

There is some $t_1 < 1$ at which y_2' is finite. Since y_2' goes from a finite value to infinity, we can find some t_2 at which it is both large and increasing. It follows that $y_2'' > 0$ at t_2. Now look at y_2''' at t_2, its first term is clearly positive. Its second term is positive because y_2' is large, implying $y_1'' > 0$. Furthermore, $y_2'' > 0$ at t_2 implies $y_1' > 3/2$, therefore the third term is also positive at t_2. We conclude that y_2'' is monotonically increasing not only at t_2 but at all $t > t_2$, which implies $y_1' > 3/2$ for all $t > t_2$. This contradicts $y_1' \downarrow 0$, given $y_2 < 1$ and $y_1 \to 1$ as $t \to 1$, from its definition $y_1' = (1 - y_1)/(t - y_2)$. QED

Lemma 7. *If for $y \in S$, $\tilde{y}_2 \leqslant \tilde{y}_1 = 1$ then $\tilde{y}_2 = \tilde{y}_1 = 1$.*

Proof. Consequence of *Lemma 6*.

Now we prove the main Theorem on the existence and uniqueness of equilibrium.

Proof. Above we showed that a separation point s exists between the initial points for S_T (that lie to the left of s) and the initial points for S_C (that lie to the right of s). Clearly, s is in the closures of both intervals $(0, s)$ and $(s, 1]$. From *Lemmas 2* and *3*, y^s is a limit solution for both S_T and S_C, and $\hat{y}^s$ is a limit solution for both $\hat{S}_T$ and $\hat{S}_C$. Suppose y^s does not solve (BVP). Specifically, assume $\tilde{y}_1^s = \delta < 1$ for some $\delta > 0$, then clearly y^s does not solve (BVP). If $\hat{y}^s$ is in $\hat{S}_T$, let $\tilde{s}$ be its termination point. Now ask whether there exists a sequence $\tilde{y}_1^{t_n}$ in $(0, 1]$ that corresponds to the sequence $\hat{y}^{t_n}$ in $\hat{S}_C$ such that $(1, \tilde{y}_1^{t_n} \to (\tilde{s}, \tilde{y}_1^s)$. The answer is clearly ''No'' since $\tilde{s} = \tilde{y}_1^s = \delta < 1$. Therefore such a sequence does not exist. This violates the connectedness of $\hat{S}$.

If $\hat{y}^s$ is not in $\hat{S}_T$ then by definition it is in $\hat{S}_C$, with $\tilde{s} = 1$. We ask whether there is a sequence $(\tilde{t}_n, \tilde{y}_1^{t_n})$ in $(0, 1] \times (0, 1]$ that corresponds to the sequence $(\tilde{t}_n, \hat{y}^{t_n})$ in $(0, 1] \times \hat{S}_T$ such that $(\tilde{t}_n, \tilde{y}_1^{t_n}) \to (1, \tilde{y}_1^s)$. The answer is ''No'', since $\tilde{t}_n = \tilde{y}_1^{t_n}$ and $\tilde{y}_1^s = \delta < 1$. Therefore such a sequence does not exist. Again, this violates the connectedness of $\hat{S}$.

We conclude that the assumption was wrong and $\tilde{y}_1^s = 1$. From *Lemma 7*, $\tilde{y}_2^s = 1$. Moreover, at $t = 1$, $(y_1^s)' = 3/2$ and $(y_2^s)' = 2$, using l'Hôpital's Rule.[36] Therefore $(y^s)(1) = f(1, y^s(1))$ with $y^s(1) = (1, 1)$ is bounded, and $y^s(1)$ is well-defined. Uniqueness of y^s, i.e. uniqueness in (BVP), follows from uniqueness in (IVP) (*Remark 1*). Setting $t_* = s$ completes the proof of the main Theorem.

Appendix B

The numerical solution

This appendix contains a *Mathematica* program to numerically solve and graphically display any system of two ordinary differential equations $g_1(t)$ and $g_2(t)$, whose derivatives are represented by functions $f_1(t, g_1(t), g_2(t))$ and $f_2(t, g_1(t), g_2(t))$ respectively. The estimation technique is ''explicit fourth-order Runge–Kutta.'' For further details see Ixaru (1984) pp. 133–34 and Ascher et al. (1995) pp. 68–72 and 210–17. We tried uniform step sizes ranging from order of 10^{-3} to 10^{-5}, the solutions of the boundary value problem identified by using different step sizes were reasonably close to each other. A fourth-order[37] Runge–

[36] Taking the derivative of each numerator and denominator separately, $y_1' = -y_1'/(1 - y_2')$ and $y_2' = -y_2'/2(1 - y_1')$ gives the result.

[37] A q^{th}-order Runge–Kutta method is a numerical solution algorithm which uses the values of the functions and their derivatives at point t^n to evaluate the value of the function at point $t^n + h$ (where h is the step size) by running q consecutive approximations and taking their weighted average at each step. Since the approximation at $t^n + \mathrm{h}$ depends on the data at t^n only, Runge–Kutta is a one-step method. For $q = 1$, the method is identical to the Euler algorithm (linear extrapolation).

Kutta method with a uniform step size is consistent[38] of order 4, the maximum order of consistency[39] attainable by Runge–Kutta.[40]

The program logic

1. At an arbitrary initial point the values of the functions and the derivatives are known analytically. Let this point be t^n, and let $g_1(t^n)=g_1^n$ and $g_2(t^n)=g_2^n$. If there is any other function p related to $g_1(t)$ and $g_2(t)$ that needs to be evaluated (e.g. $p=$expected price), let $p(t^n,\ g_1^n,\ g_2^n)=p^n$ be its initial value. Defining h as step size
2. Estimate: $k_1=hf_1(t^n,\ g_1^n,\ g_2^n)$, $l_1=hf_2(t^n,\ g_1^n,\ g_2^n)$, the increments to g_1 and g_2 by linear extrapolation, using the derivatives at the initial point.
3. Estimate: $k_2=hf_1((t^n+h/3),\ (g_1^n+k_1/3),\ (g_2^n+l_1/3))$, $l_2=hf_2((t^n+h/3),\ (g_1^n+k_1/3),\ (g_2^n+l_1/3))$, repeat starting at initial value plus a third of the first estimate, at one-third of the step length.
4. Estimate: $k_3=hf_1((t^n+2h/3),\ (g_1^n-k_1/3+k_2),\ (g_2^n-l_1/3+l_2))$, $l_3=hf_2((t^n+2h/3),\ (g_1^n-k_1/3+k_2),\ (g_2^n-l_1/3+l_2))$, again average now first two estimates, at two-thirds of the step length.
5. Estimate: $k_4=hf_1((t^n+h),\ (g_1^n+k_1-k_2+k_3),\ (g_2^n+l_1-l_2+l_3))$, $l_4=hf_2((t^n+h),\ (g_1^n+k_1-k_2+k_3),\ (g_2^n+l_1-l_2+l_3))$, averaging first three estimates, at the end of the step.
6. Evaluate the functions g_1 and g_2 at the end of the step (at $t^m=t^n+h$) as the value in the beginning of the step (at t^n) plus a weighted average of the first four step estimates $k_1,\ldots,k_4$ or $l_1,\ldots,l_4$. That is, $g_1^m=g_1(t^m)=g_1^n+(k_1+3k_2+3k_3+k_4)/8$ and $g_2^m=g_2(t^m)=g_2^n+(l_1+3l_2+3l_3+l_4)/8$.
7. Evaluate, for further use, $p^m=p^n+p(t^m,\ g_1^m,\ g_2^m)$.
8. Go to the beginning of the algorithm, repeat the steps substituting t^m, g_1^m, g_1^m for t^n, g_1^n, g_2^n, $n=0,\ 1,\ 2,\ldots,\ m=n+1$.
9. Repeat until the maximum number of steps is reached.
10. Save consecutive pairs $(t^n,\ g_1^n)$, $(t^n,\ g_2^n)$ in lists list_1, list_2, later, plot these lists.
11. Print the last values obtained before stopping, including the final value of p. If the solutions have gone over the 45°-line or they are not ‘‘close enough’’ to the

[38] In numerical solution parlance, a numerical solution method is *consistent of order p* if the *local truncation error* (the difference between the true solution and the estimated solution at one step only) is at most of order h^p where h is the step size. In contrast, a method is *convergent of order p* if the *global error* (accumulated local errors) is at most of order h^p. For one-step methods, including Runge–Kutta, a method is *consistent* if and only if it is *convergent* (Ascher et al., 1995, p. 71). In general, the global error is at most of order h^{p-1} if the local order is at most of order h^p. This conveniently provides an upper bound on the size of the truncation error in our estimates.

[39] That is, a Runge–Kutta algorithm that uses $q>4$ separate approximations at each step is consistent of order 4 only (Ixaru, 1984, p. 135).

[40] For estimation algorithms alternative to ours, see Riley and Li (1997) and Bajari (1996).

boundary values $g_1(1)=g_2(1)=1$ then change the value of the starting point t^0 (but keep the starting values of g_1^0, g_2^0, p^0 because they must hold at any starting point) by moving it to the right or to the left on the [0, 1] interval. In our problem these initial values were all zero.

The program

```
SRKStep[ {N_,k_}, {tn_,g1n_,g2n_,pn_}, h_]:=                                   \
   Block[ {K1,K2,K3,K4,L1,L2,L3,L4},                                           \
         f1 = (1-g1)/((N-k)(t-g2));                                            \
         f2 = ((N-k)(t-g2)-(N-k-1)(t-g1))(1-g2)/(k(N-k)(t-g2)(t-g1));          \
         P = t * ( k (1-g2)^(k-1) (1-g1)^(N-k) f2  +
            (N-k)(1-g2)^k (1-g1)^(N-k-1) f1 );                                 \
       K1=h f1 /.{t->tn,g1->g1n,g2->g2n};                                      \
       L1=h f2 /.{t->tn,g1->g1n,g2->g2n};                                      \
       K2=h f1 /.{t->tn+h/3,g1->g1n+K1/3,g2->g2n+L1/3};                        \
       L2=h f2 /.{t->tn+h/3,g1->g1n+K1/3,g2->g2n+L1/3};                        \
       K3=h f1 /.{t->tn+2h/3,g1->g1n-K1/3+K2,g2->g2n-L1/3+L2};                 \
       L3=h f2 /.{t->tn+2h/3,g1->g1n-K1/3+K2,g2->g2n-L1/3+L2};                 \
       K4=h f1 /.{t->tn+h,g1->g1n+K1-K2+K3,g2->g2n+L1-L2+L3};                  \
       L4=h f2 /.{t->tn+h,g1->g1n+K1-K2+K3,g2->g2n+L1-L2+L3};                  \
       tm=tn+h; g1m=g1n+(K1+3K2+3K3+K4)/8; g2m=g2n+(L1+3L2+3L3+L4)/8;          \
       pm = pn + (P /.{g1->g1m,g2->g2m,t->tm}) * h;                            \
         AppendTo[list1,{tm, g1m}];                                            \
         AppendTo[list2,{tm, g2m}];                                            \
       {tm, g1m, g2m, pm}                                                      \
   ];
SRungeKutta[ {N_,k_}, {t0_,g10_,g20_,p0_}, h_, n_]:=                           \
   ( list1={{t0,g10}}; list2={{t0,g20}}; list3={{t0,p0}};                      \
      Nest[SRKStep[ {N,k}, #, h]&, {t0,g10,g20,p0}, n]                         \
   );
SRKGraph[ {N_,k_}, {t0_,g10_,g20_,p0_}, h_, n_]:=                              \
   ( SRungeKutta[{N,k},{t0,g10,g20,p0},h,n];                                   \
        LP1=ListPlot[list1,PlotJoined->True, Frame->True];                     \
        LP2=ListPlot[list2,PlotJoined->True, Frame->True];                     \
        Show[LP1,LP2,AxesOrigin->{t0,g10}]; Print["E[p]=",pm];                 \
            Print["{tm, g1m, g2m} = ", {tm, g1m, g2m}];          \
  );
```

Appendix C

Symmetric multiple draw auctions

In this appendix we show that the symmetric profit function for N draws and R firms, each with $k=N/R\geqslant 2$ cost draws is identical to the symmetric profit function for $N-k+1$ firms, each with a single cost draw.

First we will show that the multiple cost draw leads to an equivalent expected profit function to a corresponding single cost draw game. Firm i's expected profits in the multidraw game as a function of its price $\tilde{p}_i$, for any value of its lowest cost draw $\tilde{c}_i$, is

$$\tilde{\pi}_i(\tilde{p}_i) = \text{Prob}\left\{ \tilde{p}_i < \min_{j \neq i} \tilde{P}_j(\tilde{c}_j) \right\} [\tilde{p}_i - \tilde{c}_i].$$

By symmetry, the function $\tilde{P}_j \equiv \tilde{P}$ for all j. Exploiting $\tilde{P}$ being positively sloped in c and defining its inverse as $\tilde{c}(\tilde{p})$

$$\tilde{\pi}_i(\tilde{p}_i) = \text{Prob}\left\{ \tilde{c}(\tilde{p}_i) < \min_{j \neq i} \tilde{c}_j \right\} [\tilde{p}_i - \tilde{c}_i].$$

Denote i's rivals as $-i$, their lowest cost as $\breve{c}_{-i}$ with N total cost draws, k by firm i, then the distribution of rivals' minimum cost is

$$\breve{F}^{(1)}_{-i}(\breve{c}_{-i}) = [1 - F(\breve{c}_{-i})]^{N-k}$$

so

$$\tilde{\pi}_i(\tilde{p}_i) = [1 - F(\tilde{c}(\tilde{p}_i))]^{N-k} [\tilde{p}_i - \tilde{c}_i].$$

where $F \equiv U[0, 1]$, the uniform distribution over [0, 1].

We now turn to the game where each player has a single cost draw. Recall in this game, if there are N' firms (and hence draws) the first order conditions are given by

$$\pi_i(p) = \text{Prob}\left\{ c(p) < \min_{j \neq i} c_j \right\} [p - c_i] = [1 - F^{(1)}_{-i}(c(p))][p - c_i]$$

where $F^{(1)}_{-i}(c) = 1 - [1 - F(c)]^{N'-1}$. Again, $F \equiv U[0, 1]$. So if one selects $N' = N - k + 1$ the two first order conditions are identical. What this implies is that the best response function used by all R firms, each with k cost draws where $N = Rk$, is identical to the best response function used by all $N - k + 1$ firms in the symmetric single cost draw model.

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